DISTANCE FUNCTIONS, APPROXIMATE SOURCE CONDITIONS AND THE INTERPLAY OF SMOOTHNESS IN REGULARIZATION THEORY

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ABSTRACT

In this paper, we present some new results on convergence rates in regularization of linear ill-posed operator equations in a Hilbert space setting. Such equations occur in the modelling of linear inverse problems and in the context of nonlinear inverse problems when the nonlinear problems are solved iteratively by linearization. The focus is on Tikhonov regularization and on the interplay of smoothness conditions for the solution and the degree of ill-posedness of the problem. We consider approximate source conditions based on distance functions that measure the violation of general source conditions generated by an index function that works as benchmark. Estimates for the regularization error are derived based on bounds for the distance functions. In particular, we give some explicit error bounds if there is a link between the solution smoothness and the forward operator of the inverse problem in form of a range inclusion.

1. INTRODUCTION

In Applied Mathematics the past twenty years were characterized by a growing interest in the solution of Inverse Problems. In particular, the computerized simulation (forward computation) of complex processes in wide fields of natural sciences, engineering and finance requires the determination of parameter functions, which are necessary for a serious modelling of the process. Moreover the knowledge and precision of those functions essentially influence the chances and limitations of computer simulation (cf. [2], [6]). For example, heat transfer functions depending on time and geometry play an important role for computer models using the heat equation. On the other hand, in finance volatility functions depending on asset price and maturity are crucial for the computation of prices for complex financial derivatives written on the asset (cf. [7]). Such functions frequently cannot be observed or measured directly. They have to be determined by exploiting indirect measurements or observations and solving corresponding inverse problems. In most cases, the arising inverse problems are ill-posed. That means, their solutions fail to be stable with respect to the measured data. As a consequence a regularization approach is required for the stable approximate solution of the inverse problems. Hence, the regularization theory (cf. [4], [8], [17], [24]) and also their excellent new and recent results are interesting not only from a theoretical point of view, but also for the practical use in all applications where ill-posedness occurs.

In this paper, let X and Y be infinite dimensional separable Hilbert spaces, where in both spaces the symbols $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ designate the inner products and the generic norms including corresponding operator norms, respectively. By $F: D(F) \subset X \to Y$ we denote the forward operator of a causality under consideration, i.e., the continuous mapping defined on the domain $D(F)$ that assigns the parameter function $x \in X$ a function $y \in Y$ for which data are available. Then the inverse problem can be written as an operator equation

$$F(x) = y\,. \tag{1}$$

The ill-posedness of equation (1) corresponds with the fact that the inverse operator F^{-1} whenever it exists is not continuous. We have to distinguish linear inverse problems (occurring e.g. in [22]) and nonlinear inverse problems (e.g. in [23]) in dependence of the linearity or nonlinearity of the forward operator F. If this operator is a linear bounded one, then we will denote it by the symbol A and we then consider linear operator equations

$$Ax = y \tag{2}$$

modelling that situation. Here we assume the operator $A: X \to Y$ to be injective such that the solution $x_0 \in X$ of (2) is well-defined for $y \in \mathcal{R}(A)$, where $\mathcal{R}(A)$ denotes the range of the operator A. Then ill-posedness for linear equations (2) occurs if A^{-1} is unbounded or in other words if the range $\mathcal{R}(A)$ is not closed.

The Tikhonov regularization method on which we focus in this paper provides us with stable approximate solutions for linear and nonlinear ill-posed equations. Also convergence and convergence rates results are presented in the literature for the equations (1) and (2) (see, e.g., [4]). Since the convergence of regularized solutions may be arbitrarily slow, for obtaining convergence rates it is necessary that so-called source conditions are fulfilled for the solution $x_0 \in X$ of the ill-posed equation (cf. [4], [21]). In the linear case, general source conditions have the form

$$x_0 = \varphi(A^*A)\, v \tag{3}$$

with $v \in X$ and some index function φ defined on the spectrum $[0,a]$ of the operator A^*A with $a := \|A^*A\|$. A nonnegative function φ is called index function if it is strictly increasing and continuous with $\varphi(0) = 0$.

Now it is an interesting problem to formulate in a precise manner cross-connections between source conditions (3) and the smoothness of x_0 in the sense of a condition

$$x_0 = G\, w \tag{4}$$

with some $w \in X$ and a given self-adjoint bounded linear operator $G : X \to X$ having a nonclosed range $\mathcal{R}(G)$ and its spectrum in $[0,b]$ with $b := \|G\|$. We note that the forward operator A and G can be completely independent and moreover that the interplay under consideration is closely connected with the degree of ill-posedness of (2) (cf. [1], [5], [14], [20]). Below we give some answers to the interplay problem (for more details see also [3]), but in the moment only for the linear case (2) avoiding additional difficulties which occur in view of the local change of properties in the nonlinear case (cf. [9], [13]). An adapted generalization of the ideas and results given below to nonlinear ill-posed equations (1) is forthcoming work of the author and coauthors.

2. GENERAL SOURCE CONDITIONS AND CONVERGENCE RATES

As obvious in models with deterministic noise and noise level $\delta > 0$ let be given data $y^\delta \in Y$ approximating $y \in \mathcal{R}(A)$ such that $\|y^\delta - y\| \le \delta$. Moreover, the stable approximate solutions of the linear operator equation (2) will be calculated by the Tikhonov regularization, where we distinguish regularized solutions

$$x_\alpha \;=\; (A^*A + \alpha I)^{-1}\, A^*\, y$$

in the case of noise-free data and

$$x_\alpha^\delta \;=\; (A^*A + \alpha I)^{-1}\, A^* y^\delta$$

in the case of noisy data. Here, for fixed A and x_0, we focus on the noise-free error function

$$f(\alpha) \;:=\; \|x_\alpha - x_0\| \;=\; \|\alpha\,(A^*A + \alpha I)^{-1}\, x_0\| \tag{5}$$

depending on the regularization parameter $\alpha > 0$. Taking into account the noise level δ this function determines the total regularization error of the Tikhonov regularization

$$e(\alpha,\delta) \;:=\; \|x_\alpha^\delta - x_0\| \le \|x_\alpha - x_0\| + \|x_\alpha^\delta - x_\alpha\|$$

in case of noisy data with the well-known estimate

$$e(\alpha,\delta) \;\le\; f(\alpha) + \frac{\delta}{2\sqrt{\alpha}}. \tag{6}$$

For sufficiently small positive α we search for estimates of the error function of form

$$f(\alpha) = \|\alpha\,(A^*A + \alpha I)^{-1}\, \varphi(A^*A)\, w\| \le K\varphi(\alpha)\|v\| \tag{7}$$

with some $K \ge 1$. From the literature (cf. [19], [1]) we have the following proposition.

Proposition 2.1. *Let x_0 satisfy a general source condition (3) with an index function $\varphi(t)$ $(0 \le t \le a)$. If (a) $\varphi(t)/t$ is monotonically decreasing on $(0,a]$, or (b) $\varphi(t)$ is concave on $[0,a]$, then (7) holds with $K = 1$. If there exists a $\hat{t} \in (0,a)$ such that (c) $\varphi(t)/t$ is monotonically decreasing on $(0,\hat{t}\,]$ or (d) $\varphi(t)$ is concave on $[0,\hat{t}\,]$, then (7) is true with $K = \varphi(a)/\varphi(\hat{t}\,)$.*

If an index function φ satisfies one of the requirements (a) – (d) in Proposition 2.1, then φ is a qualification of Tikhonov's regularization method in the sense of the papers [18] and [19]. On the other hand, an inequality (7) implies estimates of form

$$e(\alpha,\delta) \;\le\; K\varphi(\alpha)\,\|w\| + \frac{\delta}{2\sqrt{\alpha}} \quad (0 < \alpha \le a) \tag{8}$$

for the total regularization error. Then for sufficiently small $\overline{\delta} > 0$ we find a constant $\tilde{K} > 0$ such that

$$e(\alpha(\delta),\delta) \le \tilde{K}\, f(\Theta^{-1}(\delta)) \qquad (0 < \delta \le \overline{\delta}), \tag{9}$$

where with f also

$$\Theta(\alpha) := \sqrt{\alpha}\, f(\alpha)$$

is an index function and for the a priori choice of the regularization parameter the uniquely determined solution $\alpha(\delta) > 0$ of the equation $\Theta(\alpha) = \delta$ is taken. Note that (9) yields the function $f(\Theta^{-1}(\delta))$ as a convergence rate for the Tikhonov regularization. Under weak additional assumptions this rate is order optimal (see [19]). On the other hand, it is evident that the optimal rate is completely determined by the noise-less error function f and it is sufficient to study the behaviour of f. However, for fixed A and x_0 the function φ in (3) is not well-defined. For example, the same x_0 may satisfy a source condition (3) both for an index function of power-type $\varphi(t) = t^\nu$ with some exponent $\nu > 0$ and for an index function of logarithmic type $\varphi(t) = 1/(\log(1/t))^\mu$ for some exponent $\mu > 0$ (cf. [16]).

3. APPROXIMATE SOURCE CONDITIONS AND DISTANCE FUNCTIONS

If, for given index function φ, the solution x_0 fails to satisfy a source condition (3), we can use an alternative approach for finding estimates of the form (7) and hence (8) and consequently convergence rates for the Tikhonov regularization. In this context, we exploit the fact that any solution x_0 of (2) satisfies (3) in an approximate manner by considering distance functions

$$d_\varphi(R) := \inf\{\|x_0 - \varphi(A^*A)\,w\| \,:\, w \in X,\ \|w\| \le R\}$$

for $R \ge 0$ that measure for x_0 the violation of the source condition (3). The index function φ has the character of a benchmark function. By using general such benchmark functions we extend the corresponding results, which with focus on the special case $\varphi(t) = \sqrt{t}$ were published in the recent papers [10], [11] and [15].

Evidently, for every $x_0 \in X$ the nonnegative distance function $d_\varphi(R)$ depending on the radius $R \in [0, \infty)$ is well-defined and nonincreasing with $\lim_{R\to\infty} d_\varphi(R) = 0$ as a consequence of the injectivity of $\varphi(A^*A)$ and $\overline{\mathcal{R}(\varphi(A^*A))} = X$. Note that the injectivity of A implies for any index function φ the injectivity of $\varphi(A^*A)$ The distance function $d_\varphi(R)$ expresses the behaviour of x_0 with respect to the benchmark condition (3). There are two cases: Case (a) with $x_0 \notin \mathcal{R}(\varphi(A^*A))$ and $d_\varphi(R) > 0$ for all $R \ge 0$ as well as case (b) with $x_0 \in \mathcal{R}(\varphi(A^*A))$ implying for some $R_0 > 0$ the situation $d_\varphi(R) > 0$ $(0 \le R < R_0)$ and $d_\varphi(R) = 0$ $(R \ge R_0)$. Only the case (a) is of interest here. For that case one can show based on the Lagrange multiplier method used in the proof of Lemma 2.5 in [10] that $d_\varphi(R)$ is a strictly decreasing function for $R \in (0, \infty)$ and consequently that $d_\varphi(1/t)$ is an index function for $t > 0$. Hence

$$\theta(t) := t\, d_\varphi(1/t) \ \ (t > 0), \qquad \theta(0) := 0 \tag{10}$$

is an index function on every interval $[0, \bar{t}\,]$. Note that this function θ is fundamental for the use of approximate source conditions in the paper [12] that tries to generalize such ideas to general linear regularization schemes. The following lemma was proven in [3] (see also [11]).

Lemma 3.1. *Let $\varphi(t)$ $(0 \le t \le a)$ be an index function that satisfies one of the requirements (a) – (d) in Proposition 2.1 with the corresponding constant $K \ge 1$. Then we obtain the error estimate for the Tikhonov regularization*

$$f(\alpha) \,=\, \|x_\alpha - x_0\| \,\le\, d_\varphi(R) + K\varphi(\alpha)\,R \tag{11}$$

for all $\alpha > 0$ and $R \ge 0$.

This lemma is basic for the subsequent theorem.

Theorem 3.2. *Let the assumptions of Lemma 3.1 hold. Moreover let*

$$x_0 \notin \mathcal{R}(\varphi(A^*A))\,. \tag{12}$$

Then for sufficiently small $\alpha > 0$ we have an error estimate

$$f(\alpha) = \|x_\alpha - x_0\| \,\le\, (K+1)\,\frac{\varphi(\alpha)}{\theta^{-1}(\varphi(\alpha))} \tag{13}$$

for the Tikhonov regularization and hence a rate $f(\alpha) = \mathcal{O}\left(\frac{\varphi(\alpha)}{\theta^{-1}(\varphi(\alpha))}\right)$ as $\alpha \to 0$ with the index function θ from (10) for this method and for the solution $x_0 \in X$.

Proof: We use the estimate (11), which is valid for all $R > 0$, and equate the terms $d_\varphi(R)$ and $R\varphi(\alpha)$. By setting $t := 1/R$ this is equivalent to $\theta(t) = \varphi(\alpha)$. For $\alpha > 0$ small enough there is some $t = t(\alpha) = \theta^{-1}(\varphi(\alpha))$ such that this equation is fulfilled and we find (13) from (11) taking into account that both φ and θ and also θ^{-1} are index functions. This proves the theorem. ∎

Remark 3.3. Note that the rate $\frac{\varphi(\alpha)}{\theta^{-1}(\varphi(\alpha))}$ is slower than the rate $\varphi(\alpha)$ prescribed by the benchmark function, since $\lim_{t\to 0} 1/\theta^{-1}(t) = \infty$. This is a consequence of the assumption (12). Moreover, we state that the formulae (11) and (13) remain true if we replace $d_\varphi(R)$, also in formula (10), by a majorant which is a strictly decreasing function of R tending to zero as $R \to \infty$.

4. SPECIFIC RESULTS FOR POWER-TYPE FUNCTIONS AND COMPACT FORWARD OPERATORS

Depending on the relative smoothness of x_0 with respect to the forward operator A the distance functions $d_\varphi(R) \to 0$ as $R \to \infty$ may have very slow (logarithmic) decay rates as $R \to \infty$ as one extremal case, or they may have very fast (exponential) decay rates the second extremal case. The consequences of both extremal situations for convergence rates of Tikhonov regularization are outlined for the benchmark functions $\varphi(t) = \sqrt{t}$ in [10]. Such considerations can easily be generalized to a general index function φ. In this section we present a detailed discussion of the moderate case of a decay of $d_\varphi(R)$ characterized by a power-type rate.

Theorem 4.1. *We choose as benchmark function φ for the distance function $d_\varphi(R)$ a power-type function*

$$\varphi(t) = t^\nu \qquad (0 \le t \le a) \tag{14}$$

with exponent $0 < \nu \le 1$. Moreover, we assume that the solution x_0 of equation (2) satisfies the condition (12). Then a power-type decay rate of the distance function as

$$d_\varphi(R) \le \frac{C}{R^{\frac{\eta}{\nu-\eta}}} \qquad (\underline{R} \le R < \infty) \tag{15}$$

with $0 < \eta < \nu \le 1$ for some positive constants C and $\underline{R}$ implies an estimate for the regularization error of form

$$f(\alpha) = \|x_\alpha - x_0\| \le \widehat{C}\,\alpha^\eta \qquad (0 < \alpha \le \overline{\alpha}) \tag{16}$$

with some positive constants $\widehat{C}$ and $\overline{\alpha}$.

Proof: Noting that φ is concave, we can immediately apply Theorem 3.2, where in the sense of Remark 3.3 $d_\varphi(R)$ is replaced by its majorant $C\,R^{-\frac{\eta}{\nu-\eta}}$. This yields $\theta(t) = C\,t^{\frac{\nu}{\nu-\eta}}$, for sufficiently small positive t then $\theta^{-1}(t) = \widetilde{C}\,t^{\frac{\nu-\eta}{\nu}}$, and for sufficiently small $\alpha > 0$ hence $\theta^{-1}(\varphi(\alpha)) = \widetilde{C}\,\alpha^{\nu-\eta}$ and finally $\varphi(\alpha)/\theta^{-1}(\varphi(\alpha)) = \widehat{C}\,\alpha^\eta$ with corresponding constants $\widetilde{C}$ and $\widehat{C}$. This proves the theorem. ∎

We give some comments on the above theorem. First we remark that the exponent $\frac{\eta}{\nu-\eta}$ in formula (15) attains all positive values if η varies through the whole the open interval $(0,\nu)$. Secondly, we note that it is evident from Proposition 2.1 that an error estimate $f(\alpha) = \mathcal{O}(\alpha^\eta)$ as obtained with formula (16) in Theorem 4.1 also occurs if x_0 satisfies a source condition $x_0 = (A^*A)^\eta\, v$ with $v \in X$ for power-type source function with exponent $0 < \eta < \nu$. So it seems to be of some interest to answer the question whether $x_0 \in \mathcal{R}((A^*A)^\eta)$ also implies a decay rate of form (15) for the distance function. We prove such a converse result for the case of compact operators A.

Theorem 4.2. *Let be given a benchmark function φ of form (14). Moreover, we suppose that the operator A is compact and that the smoothness of the solution x_0 of equation (2) is characterized on the one hand by a source condition*

$$x_0 = (A^*A)^\eta\, v \quad (v \in X,\ 0 < \eta < \nu \le 1) \tag{17}$$

and on the other hand by (12). Then we have a decay rate (15) for the distance function d_φ.

Proof: We suppose that the compact operator A has the ordered singular values $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_{i-1} \ge \sigma_i \ge \ldots$, where $\lim_{i\to\infty} \sigma_i = 0$ and $\{u_i\}_{i=1}^\infty \subset X$ is a complete orthonormal system of eigenelements such that $A^*A\,u_i = \sigma_i\,u_i$ $(i = 1, 2, \ldots)$. Now $x_0 \in \mathcal{R}((A^*A)^\eta)$ obtained from (17) is equivalent to

$$\sum_{i=1}^\infty \frac{\langle x_0,\, u_i\rangle^2}{\sigma_i^{4\eta}} = \widetilde{K} < \infty$$

(cf. [4, Proposition 3.13]). Then by (14) and (17) we can use the Lagrange multiplier method for finding an explicit expression for $d_\varphi(R)$ as

$$\begin{aligned} d_\varphi(R) &= \|\lambda\,\left[(A^*A)^{2\nu} + \lambda\,I\right]^{-1} (A^*A)^\eta\, v\| \\ &\le \|\lambda\,\left[(A^*A)^{2\nu} + \lambda\,I\right]^{-1} (A^*A)^\eta\|\,\|v\| \le \lambda^{\frac{\eta}{2\nu}}\,\|v\|, \end{aligned} \tag{18}$$

where $\lambda = \lambda(R)$ is the uniquely determined solution of equation

$$R^2 = \|\left[(A^*A)^{2\nu} + \lambda I\right]^{-1} (A^*A)^\nu\, x_0\|^2\,. \tag{19}$$

More precisely, the estimate (18) is obtained from spectral theory by

$$\|\lambda\,\left[(A^*A)^{2\nu} + \lambda\,I\right]^{-1} (A^*A)^\eta\| = \sup_{0<s\le a} \frac{\lambda\,s^\eta}{s^{2\nu}+\lambda}$$

$$\le \sup_{0<s\le a} \left(\frac{\lambda^{(1-\frac{\eta}{2\nu})}\,[s^{2\nu}]^{\frac{\eta}{2\nu}}}{s^{2\nu}+\lambda}\right) \lambda^{\frac{\eta}{2\nu}} \le \lambda^{\frac{\eta}{2\nu}}\,,$$

since we have $\frac{\lambda^{(1-\frac{\eta}{2\nu})}\,[s^{2\nu}]^{\frac{\eta}{2\nu}}}{s^{2\nu}+\lambda} \le 1$ as a consequence of Young's inequality. In order to derive (15) from (18), we use a majorant for the right-hand side of (19). Indeed we get

$$R^2 = \sum_{i=1}^\infty \frac{(\sigma_i^2)^{2\nu}\,\langle x_0,\, u_i\rangle^2}{[(\sigma_i^2)^{2\nu}+\lambda]^2}$$

$$\le \sum_{i=1}^\infty \frac{\langle x_0,\, u_i\rangle^2}{\sigma_i^{4\eta}} \left(\frac{\lambda^{(1-\frac{\eta}{\nu})}\,[(\sigma_i^2)^{2\nu}]^{(1+\frac{\eta}{\nu})}}{[(\sigma_i^2)^{2\nu}+\lambda]^2}\right) \lambda^{(\frac{\eta}{\nu}-1)}$$

$$\le \widetilde{K}\,\lambda^{(\frac{\eta}{\nu}-1)}\,,$$

since we again have $\frac{\lambda^{(1-\frac{\eta}{\nu})}\,[(\sigma_i^2)^{2\nu}]^{(1+\frac{\eta}{\nu})}}{[(\sigma_i^2)^{2\nu}+\lambda]^2} \le 1$ due to Young's inequality. Then for all $R > 0$ it holds the inequality $\lambda \leq \tilde{\lambda}$ with λ solving (19) and $\tilde{\lambda}$ solving the equation $R^2 = \widetilde{K}\,\lambda^{(\frac{\eta}{\nu}-1)}$, which yields $\tilde{\lambda} = \widehat{K}\,R^{\frac{2\nu}{\eta-\nu}}$ with some positive constant $\widehat{K}$. If we exploit this result for further estimation from above of $d_\varphi(R)$ based on (18) we find $d_\varphi(R) \leq C\,R^{\frac{-\eta}{\eta-\nu}}$ for some constant $K > 0$. This, however, can be rewritten as (15) and proves the theorem. ■

5. SOLUTION SMOOTHNESS AND THE DEGREE OF ILL-POSEDNESS EXPRESSED BY RANGE INCLUSIONS

Provided that the chosen benchmark function φ is an index function on $[0,a]$ that satisfies one of the requirements (a) – (d) in Proposition 2.1, then based on Lemma 3.1 and Theorem 3.2 convergence rates of Tikhonov regularization can be found for solutions x_0 if majorants of the distance function $d_\varphi(R)$ are available. For a given self-adjoint bounded linear operator $G: X \to X$ with nonclosed range $\mathcal{R}(G)$ and with spectrum in the interval $[0,b]$ such majorant functions will be derived in this section under the following three assumptions. We assume for index functions ρ_1, ρ_2 defined on $[0,b]$ the range inclusion

$$\mathcal{R}(\rho_1(G)) \subset \mathcal{R}(\varphi(A^*A))\,, \tag{20}$$

the smoothness condition

$$x_0 = \rho_2(G)\,w \quad (w \in X) \tag{21}$$

and that there is some $0 < \varepsilon \leq b$ such that

$$q(0) := 0, \quad q(t) := \left(\tfrac{\rho_1}{\rho_2}\right)(t) \quad (0 < t \leq \varepsilon) \quad \text{is an index function on } [0,\varepsilon]\,. \tag{22}$$

We note that under the assumptions stated above, in particular due to the continuity of the quotient function $q(t)$ in (22) which is positive for $t > 0$, there exists some constant $C_1 > 0$ such that

$$\sup_{\varepsilon \leq t \leq b} \left(\frac{\rho_2}{\rho_1}\right)(t) \leq C_1 \left(\frac{\rho_2}{\rho_1}\right)(\varepsilon)\,. \tag{23}$$

The study of this section is an extension of the recent results of [15]. In contrast to [15] we use general benchmark functions φ being a qualification of Tikhonov's method, noting that $\varphi(A^*A)$ is a self-adjoint bounded linear operator with nonclosed range for any index function φ whenever A is so. However, we remark that the consequences of the assumptions (20) with $\varphi(t) = \sqrt{t}$ and (21) for convergence rates of Tikhonov regularization were also discussed in [1].

It is evident that the conditions (20) – (23) represent the counterpart of the standing assumption in [15] with respect to our extension. Note that the range $\mathcal{R}(\varphi(A^*A))$ is 'large' if the decay rate of the index function $\varphi(t) \to 0$ as $t \to 0$ is 'slow' and vice versa the range is 'small' if the decay rate is 'fast'. Hence under all qualification φ of Tikhonov regularization the range is 'smallest' if $\varphi(t) = t$.

First we reformulate Lemma 2 from [15], where the operator A in the original lemma is replaced by the self-adjoint operator $\varphi(A^*A)$ in our context. Taking into account that zero is an accumulation point of the spectrum of G the proof of this and of the subsequent lemma have been done in [3] without using the compactness of G, which was crucial in the corresponding proofs of the paper [15].

Lemma 5.1. *There exists some constant $C_2 > 0$ such that the inclusion*

$$\{x \in X : x = \rho_1(G)w,\, w \in X,\, \|w\| \leq C_2 R\} \subset \{x \in X : x = \varphi(A^*A)w,\, w \in X,\, \|w\| \leq R\} \tag{24}$$

is valid for all $R > 0$.

Then we can find explicit upper bounds for the distance function $d_\varphi(R)$ by the following lemma. This is the basis for the application of Theorem 3.2 under the assumptions (20) – (22).

Lemma 5.2: *There is some $\underline{R} > 0$ such that, for $w \neq 0$ from (21),*

$$d_\varphi(R) \leq \rho_2\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2 R}{C_1\|w\|}\right)\right)\|w\| \quad (R > \underline{R})\,.$$

Note that the function $\rho_2\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}(R)\right)$ occurring in the bound of Lemma 5.2 is a strictly decreasing function for sufficiently large $R > 0$ tending to zero as $R \to \infty$. Then recalling Remark 3.3 Theorem 3.2 applies and yields the main theorem of this paper.

Theorem 5.3. *Let the assumptions (20) – (22) and the assumptions of Lemma 3.1 hold. Then we have an estimate*

$$f(\alpha) \leq (K+1)\,\max\left(\frac{C_1}{C_2},\,1\right)\,\rho_2\left(\rho_1^{-1}\left(\varphi(\alpha)\right)\right)\,\|w\| \tag{25}$$

for the noise-less error of Tikhonov regularization, which is valid for sufficiently small positive $\alpha > 0$.

Proof: Since we have a bound for $d_\varphi(R)$ from Lemma 5.2, we can make explicit the corresponding function θ from (10) and apply Theorem 3.2.

Precisely, we obtain

$$\theta(t) = t\,\rho_2\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2}{C_1\|w\|t}\right)\right)\|w\|$$

$$= \frac{C_2}{C_1}\,\rho_1\left(\left(\frac{\rho_2}{\rho_1}\right)^{-1}\left(\frac{C_2}{C_1\|w\|t}\right)\right)$$

and hence by formula (25)

$$f(\alpha) \le (K+1)\,\frac{\varphi(\alpha)}{\theta^{-1}(\varphi(\alpha))}$$

$$= (K+1)\,\rho_2\left(\rho_1^{-1}\left(\frac{C_1}{C_2}\varphi(\alpha)\right)\right)\|w\|$$

for sufficiently small positive α. Since the function $\rho_2(\rho_1^{-1}(t))/t$ is nonincreasing for sufficiently small positive t, we can further estimate

$$f(\alpha) \le (K+1)\,\max\left(\frac{C_1}{C_2},1\right)\,\rho_2\left(\rho_1^{-1}\left(\varphi(\alpha)\right)\right)\,\|w\|\,.$$

This proves the theorem. ■

Based on Theorem 5.3 one can evaluate explicit convergence rates for the Tikhonov regularization whenever the functions ρ_1 and ρ_2 are available. Examples for such situations are given in the paper [15].

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